

Exotic \mathbb{R}^4 's and positive isotropic curvature

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Abstract

We show that no exotic \mathbb{R}^4 admits a complete Riemannian metric with uniformly positive isotropic curvature and with bounded geometry. This is essentially a corollary of the main result in [Hu1], and was stated in [Hu2] without proof. In the process of the proof we also show that the diffeomorphism type of an infinite connected sum of some connected smooth n -manifolds ($n \geq 2$) according to a locally finite graph does not depend on the gluing maps used.

Key words: exotic \mathbb{R}^4 's, positive isotropic curvature, infinite connected sum

1 Introduction

In [Hu1] we proved the following result which extends [CZ] to the noncompact case.

Theorem 1.1. *Let X be a complete, connected, non-compact 4-manifold with uniformly positive isotropic curvature, with bounded geometry and with no essential incompressible space form. Then X is diffeomorphic to an infinite connected sum of \mathbb{S}^4 , \mathbb{RP}^4 , $\mathbb{S}^3 \times \mathbb{S}^1$, and /or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$.*

Note that here we use the standard smooth structures of \mathbb{S}^4 , \mathbb{RP}^4 , $\mathbb{S}^3 \times \mathbb{S}^1$ and $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$.

Now we explain the notion of infinite connected sum used here (compare [BBM] and [Hu2]). Let G be a countably infinite graph which is connected and locally finite (here we allow an edge to connect a vertex to itself, and allow more than one edge to connect two vertices (or connect one vertex to itself)), and let \mathcal{X} be a class of connected, smooth n -manifolds ($n \geq 2$). We associate an element $X_v \in \mathcal{X}$ to each vertex v of G . For each edge of G , suppose it connects the vertices v_1 and v_2 (it may be that $v_1 = v_2$), we do a connected sum of X_{v_1} and X_{v_2} (as in pp. 102-106 in [BJ]). The result is a connected, smooth n -manifold, which is called an infinite connected sum of members of \mathcal{X} according to the graph G .

The following result is stated in [Hu2] without proof. It is essentially a corollary of Theorem 1.1.

Theorem 1.2. *No exotic \mathbb{R}^4 admits a complete Riemannian metric with uniformly positive isotropic curvature and with bounded geometry.*

To prove Theorem 1.2 we also need the fact that the diffeomorphism type of an infinite connected sum of some connected smooth n -manifolds ($n \geq 2$) according to a locally finite graph does not depend on the gluing maps used. This fact is proved in Section 2. Theorem 1.2 itself is proved in Section 3.

2 Infinite connected sum

We give more details of the definition of infinite connected sum. Let G be as in the Introduction. Let $\{v_1, v_2, \dots\}$ be the set of the vertices in G . If the connected, smooth n -manifold ($n \geq 2$) X_{v_i} associated to the vertex v_i is orientable, we choose an orientation of it. For each pair (i, j) with $i \leq j$, let $\{e_{ij}^k | k = 1, 2, \dots, m_{ij}\}$ be the set of edges connecting v_i to v_j (of course, if there is no edge connecting v_i to v_j , $m_{ij} = 0$, and in this case this set is empty). To each edge e_{ij}^k we associate a pair of smooth embeddings $f_{e_{ij}^k} : \mathbb{R}^n \rightarrow X_{v_i}$ and $g_{e_{ij}^k} : \mathbb{R}^n \rightarrow X_{v_j}$ from the standard \mathbb{R}^n . We fix an orientation of the standard \mathbb{R}^n . If X_{v_i} is oriented, we let $f_{e_{ij}^k}$ ($j \geq i$, $k = 1, 2, \dots, m_{ij}$) be orientation-preserving, and let $g_{e_{pi}^l}$ ($p \leq i$, $l = 1, 2, \dots, m_{pi}$) be orientation-reversing. We assume that the images of all these embeddings are disjoint from each other.

Let D^n be the closed unit ball with center the origin in the standard \mathbb{R}^n .

For each i , let

$$Y_i := X_{v_i} \setminus (\cup_{j \geq i} \cup_{k=1}^{m_{ij}} f_{e_{ij}^k}(\frac{1}{3}D^n) \cup \cup_{p \leq i} \cup_{l=1}^{m_{pi}} g_{e_{pi}^l}(\frac{1}{3}D^n)),$$

and let Y be the infinite disjoint union $\bigsqcup Y_i$.

We define an equivalence relation \sim in Y by setting

$$f_{e_{ij}^k}(tu) \sim g_{e_{ij}^k}((1-t)u)$$

for all (i, j) with $i \leq j$, $k = 1, 2, \dots, m_{ij}$, $t \in (\frac{1}{3}, \frac{2}{3})$ and $u \in \mathbb{S}^{n-1}$. Let X be the quotient space Y / \sim . We call X the connected sum of X_{v_i} according to the graph G via $\{f_{e_{ij}^k}, g_{e_{ij}^k}\}$, and denote it by $\sharp_G X_{v_i}(f_{e_{ij}^k}, g_{e_{ij}^k})$.

Let M^n be a connected n -manifold, and $\varphi = \sqcup_{i=1}^k \varphi_i : \sqcup_{i=1}^k D^n \rightarrow M^n$ and $\tilde{\varphi} = \sqcup_{i=1}^k \tilde{\varphi}_i : \sqcup_{i=1}^k D^n \rightarrow M^n$ be two embeddings from the disjoint union of k copies of the standard n -disk. As in [BJ, Definition (10.1)] we say φ and $\tilde{\varphi}$ are compatibly oriented if either M^n is not orientable, or, for each i ($1 \leq i \leq k$), φ_i and $\tilde{\varphi}_i$ are both orientation preserving or both orientation reversing (relative to fixed orientations of D^n and M^n).

The following result is well-known, see for example Theorem 3.2 in Chapter 8 of [H].

Proposition 2.1. *Let $\varphi = \sqcup_{i=1}^k \varphi_i : \sqcup_{i=1}^k D^n \rightarrow M^n$ and $\tilde{\varphi} = \sqcup_{i=1}^k \tilde{\varphi}_i : \sqcup_{i=1}^k D^n \rightarrow M^n$ be two (smooth) embeddings from the disjoint union of k ($< \infty$) copies of the standard n -disk to a connected, smooth n -manifold M^n ($n \geq 2$). Suppose that φ and $\tilde{\varphi}$ are compatibly oriented. Then there is a diffeotopy H of M^n , which is fixed outside of a compact subset of M^n such that $H(\cdot, 1) \circ \varphi = \tilde{\varphi}$.*

(For definition of diffeotopy (or ambient isotopy), see [BJ, Definition (9.3)] and p.178 of [H].)

Proof We follow closely the proof of Theorem 3.2 in Chapter 8 of [H]. We do induction on k . The $k = 1$ case is due to Cerf and Palais (for expositions see Chapters 9 and 10 in [BJ], Chapter III in [K] and Theorem 3.1 in Chapter 8 of [H]). Suppose the result is true for $k = j$. Now we consider the case $k = j + 1$. By assumption there exists a diffeotopy \tilde{H} of M^n , which is fixed outside of a compact subset of M^n such that $\tilde{H}(\cdot, 1) \circ \varphi|_{\sqcup_{i=1}^j D^n} = \tilde{\varphi}|_{\sqcup_{i=1}^j D^n}$. (In particular, it follows that $\tilde{H}(\cdot, 1)(\varphi_{j+1}(D^n)) \subset M^n \setminus \cup_{i=1}^j \tilde{\varphi}_i(D^n)$.) Since $n \geq 2$, $M^n \setminus \cup_{i=1}^j \tilde{\varphi}_i(D^n)$ is connected. We apply the $k = 1$ case to the two embeddings

$$\tilde{H}(\cdot, 1) \circ \varphi_{j+1}, \tilde{\varphi}_{j+1} : D^n \rightarrow M^n \setminus \cup_{i=1}^j \tilde{\varphi}_i(D^n),$$

and get a diffeotopy \hat{H} of $M^n \setminus \cup_{i=1}^j \tilde{\varphi}_i(D^n)$ which is fixed outside of a compact subset of $M^n \setminus \cup_{i=1}^j \tilde{\varphi}_i(D^n)$ such that $\hat{H}(\cdot, 1) \circ \tilde{H}(\cdot, 1) \circ \varphi_{j+1} = \tilde{\varphi}_{j+1}$. Clearly \hat{H} extends to a diffeotopy of M^n which leaves $\cup_{i=1}^j \tilde{\varphi}_i(D^n)$ fixed. Then $H_t := \hat{H}_t \circ \tilde{H}_t$ is the desired diffeotopy. \square

Theorem 2.2. *The infinite connected sum $\sharp_G X_{v_i}(f_{e_{ij}^k}, g_{e_{ij}^k})$ is a connected, smooth manifold, and oriented if all X_{v_i} are oriented. Its diffeomorphism type (oriented if relevant) does not depend on the choice of embeddings $f_{e_{ij}^k}$ and $g_{e_{ij}^k}$.*

Proof The first claim can be shown as in pp. 103-104 in [BJ] and pp. 90-91 in [K]. Now we show the second claim. Suppose that to each edge e_{ij}^k we associate another pair of smooth embeddings $\tilde{f}_{e_{ij}^k} : \mathbb{R}^n \rightarrow X_{v_i}$ and $\tilde{g}_{e_{ij}^k} : \mathbb{R}^n \rightarrow X_{v_j}$ from the standard \mathbb{R}^n . If X_{v_i} is oriented, we let $\tilde{f}_{e_{ij}^k}$ ($j \geq i, k = 1, 2, \dots, m_{ij}$) be orientation-preserving, and let $\tilde{g}_{e_{pi}^l}$ ($p \leq i, l = 1, 2, \dots, m_{pi}$) be orientation-reversing. We assume that the images of all these embeddings $\tilde{f}_{e_{ij}^k}$ and $\tilde{g}_{e_{pi}^l}$ are disjoint from each other. We define \tilde{Y}_i and \tilde{Y} as before using $\tilde{f}_{e_{ij}^k}$ and $\tilde{g}_{e_{ij}^k}$. We also introduce an equivalence relation in \tilde{Y} as before using $\tilde{f}_{e_{ij}^k}$ and $\tilde{g}_{e_{ij}^k}$, and still denote it by \sim . Finally we let $\sharp_G X_{v_i}(\tilde{f}_{e_{ij}^k}, \tilde{g}_{e_{ij}^k}) := \tilde{Y} / \sim$. For each i let

$$\varphi_i := \sqcup_{j \geq i} \sqcup_{k=1}^{m_{ij}} \tilde{f}_{e_{ij}^k} \sqcup \sqcup_{p \leq i} \sqcup_{l=1}^{m_{pi}} \tilde{g}_{e_{pi}^l} : \sqcup \mathbb{R}^n \rightarrow X_{v_i}$$

and

$$\tilde{\varphi}_i := \sqcup_{j \geq i} \sqcup_{k=1}^{m_{ij}} \tilde{f}_{e_{ij}^k} \sqcup \sqcup_{p \leq i} \sqcup_{l=1}^{m_{pi}} \tilde{g}_{e_{pi}^l} : \sqcup \mathbb{R}^n \rightarrow X_{v_i}.$$

Since G is locally finite, for each i , the above $\sqcup \mathbb{R}^n$ is a finite disjoint union; we consider the finite disjoint union $\sqcup D^n$ contained in it. For each i , we can apply Proposition 2.1 to $\varphi_i|_{\sqcup D^n}$ and $\tilde{\varphi}_i|_{\sqcup D^n}$, and get a diffeotopy $H_i(\cdot, t)$ of X_{v_i} , which is fixed outside a compact subset of X_{v_i} , such that

$$\tilde{\varphi}_i|_{\sqcup D^n} = H_i(\cdot, 1) \circ \varphi_i|_{\sqcup D^n}. \quad (2.1)$$

By equation (2.1) we can define a map $F : Y = \sqcup Y_i \rightarrow \tilde{Y} = \sqcup \tilde{Y}_i$ via

$$F(x) = H_i(x, 1) \quad \text{when } x \in Y_i \text{ for some } i.$$

Clearly F is a diffeomorphism. Note that by equation (2.1) again F is compatible with the equivalence relations in Y and in \tilde{Y} . So F induces a diffeomorphism

$$\overline{F} : \sharp_G X_{v_i}(f_{e_{ij}}^k, g_{e_{ij}}^k) \rightarrow \sharp_G X_{v_i}(\tilde{f}_{e_{ij}}^k, \tilde{g}_{e_{ij}}^k).$$

□

Remark In general, if each X_{v_i} is orientable, the diffeomorphism type of the infinite connected sum $\sharp_G X_{v_i}$ may depend on the choice of the orientations of X_{v_i} . But it is easy to see that if each X_{v_i} is orientable and admits an orientation-reversing diffeomorphism, then the (unoriented) diffeomorphism type of $\sharp_G X_{v_i}$ does not depend on the choice of the orientations of X_{v_i} .

3 Proof of Theorem 1.2

Let X be a smooth 4-manifold which is homeomorphic to the standard \mathbb{R}^4 . Assume that X admits a complete Riemannian metric with uniformly positive isotropic curvature and with bounded geometry. Clearly X contains no essential incompressible space form. By Theorem 1.1, X is diffeomorphic to an infinite connected sum of \mathbb{S}^4 , \mathbb{RP}^4 , $\mathbb{S}^3 \times \mathbb{S}^1$, and /or $\mathbb{S}^3 \tilde{\times} \mathbb{S}^1$ according to a locally finite graph G . Since the fundamental group of X is trivial, the graph G must be a tree, and the smooth manifold X_v associated to any vertex v in G must be diffeomorphic to the standard \mathbb{S}^4 .

We know that any topological manifold homeomorphic to \mathbb{R}^4 has exactly one (topological) end (for definition see for example, [DK]). It follows that the tree G has exactly one topological end also. But for a locally finite graph, there is a natural bijection between its topological ends and its graph-theoretical ends, cf. [DK] and the references therein. So the tree G has only one graph-theoretical end. (There should be a more direct argument for this fact.) Now we choose a ray γ in G , which is essentially unique. Let w_0, w_1, w_2, \dots be the set of vertices along the ray γ . For each i , there are only finite vertices which can be connected to w_i via a sequence of edges not contained in the ray γ . For each i , we do connected sum of all \mathbb{S}^4 's associated to these finite vertices (including w_i), the result is diffeomorphic to the \mathbb{S}^4 associated to the vertex w_i via a diffeomorphism not affecting the part of this \mathbb{S}^4 where its connected sum with the two \mathbb{S}^4 's associated to w_{i-1} and w_{i+1} occurs. Then we see that X is diffeomorphic to an infinite connected sum of \mathbb{S}^4 's according to the ray $[0, +\infty)$ with a vertex w_i at i ($i = 0, 1, 2, \dots$) using some gluing maps.

We know that the infinite connected sum of \mathbb{S}^4 's (all with the standard orientation) according to the ray $[0, +\infty)$ using certain special gluing maps actually produces the standard \mathbb{R}^4 . (Represent the standard \mathbb{R}^4 as the union of the unit

4-ball and the closed subspaces A_i bounded by the two 3-spheres with radius i and $i + 1$ (and each with center the origin), $i = 1, 2, \dots$. Note that each A_i may be seen as \mathbb{S}^4 with two open 4-balls removed.) We also know that \mathbb{S}^4 admits an orientation-reversing diffeomorphism. So by Theorem 2.2 and the Remark following it, X is diffeomorphic to the standard \mathbb{R}^4 . \square

Remark It is interesting to see whether or not the condition ‘with bounded geometry’ in Theorem 1.2 can be removed.

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